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Quantum state discrimination with assisted entanglement

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Abstract. State discrimination is one of the key steps in quantum information tasks. This paper introduces the assisted quantum entanglement to achieve this step. For this purpose, the paper introduces some preliminary knowledge to help enter the research topic, such as the matrix rank, Kronecker product, quantum states and the calculation method of Schmidt decomposition and rank. The latter involves the Singular value decomposition in matrix factorization theory. The method of distinguishing orthogonal states in single and many-body systems is also introduced, by using the so-called positive operator-valued measurements. Then the paper proves the fact that single or multipartite system non-orthogonal states are indistinguishable regardless of whether there are auxiliary resources, which may be either separable or entangled states. The paper further illustrates the feasibility of maximum entangled states acting on two-body systems and gives corresponding examples. At the same time, the feasibility of auxiliary resources for smaller entanglement is also investigated.

Keywords: Quantum Information, State Discrimination, Entanglement.

1. Introduction

State discrimination is one of the key tasks in quantum information science. The local indistinguishability of orthogonal product states has been studied by Y. Feng and Y. Shi [1]. Next, the local distinguishability of three-qubit orthogonal states was constructed based on the methodology by Y. H. Yang, et al. [2]. The role of dimension has also paid much attention to the indistinguishability of product states [3]. The paper by Y. L. Wang, et. al also stated some methods to construct locally indistinguishable orthogonal product states for general multipartite systems [4]. The perfect local distinguishability of orthogonal maximally entangled states in canonical form with the requirement of having two copies of each state has been discussed by S. Ghosh, et. al [5]. Next, the research by H. Fan shared that linearly independent quantum states cannot be locally distinguished, but specific projecting measurements can locally distinguish subsets of maximally entangled states [6]. M. Nathanson examined the perfect distinguishability of quantum states, including orthogonal maximally entangled states and sets where perfect distinguishability is impossible [7]. N. Yu, et. al proves that there is no locally indistinguishable bipartite subspace with a qubit subsystem and demonstrates an application related to quantum channels with two Kraus operators and optimal environment-assisted classical capacity [8]. In the research by R. Duan, et. al, the lower bound of the number of members of an arbitrary basis in multipartite quantum state space can be unambiguously distinguished using local operations and classical communications (LOCC) [9]. In the research by S. Bandyopadhyay, et. al, locally distinguishable orthogonal mixed states can be characterized by their supports, leading to two types of upper bounds on their number [10]. The first depends on pure-state entanglement within the supports, while the second optimizes bounding quantities over all ensembles admissible within the density matrices' supports.

In this paper, we investigate the state discrimination assisted by entanglement. We begin by introducing the fundamental knowledge and skills used in later sections, including the matrix properties, Kronecker product, unitary matrices, orthonormal basis, Schmidt decomposition, and Schmidt rank, as well as the state discrimination of single and bipartite systems. Then we introduce the non-discrimination of non-orthogonal states, which is also the basic theory in state discrimination tasks. Further, we explore the tasks of state discrimination of bipartite systems assisted with product or entangled states. In particular, we show that three Bell states can be distinguished using a Bell state as assisted entanglement, in terms of the Schmidt decomposition of newly constructed bipartite states. We further study whether the assisted entanglement can be reduced to a smaller amount, as quantum entanglement is a valuable resource for quantum information processing.

2. Preliminaries

In this section, we review several basic terminology and facts used in this paper. In Sec. 2.1, we review the Kronecker product of two matrices. In Sec. 2.2, we introduce the Schmidt decomposition and Schmidt rank. In Sec. 2.3, we enter the preliminary work of the research topic, namely the state discrimination of single systems. Then we introduce examples of state discrimination of bipartite systems in Sec. 2.4.

2.1. Kronecker Product and Matrix operation property

Suppose A is an n×p matrix, and B is an $m \times q$ matrix. Then the Kronecker product $A \otimes B$ of A and B is the $mn \times pq$ matrix $\begin{bmatrix} a_{11}B & \cdots & a_{1p}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{np}B \end{bmatrix}$. Here are some basic properties. Firstly, the Kronecker product satisfies the associativity and distributivity, namely $A\otimes(B \otimes C) = (A \otimes B) \otimes C$ and (A+B) $\otimes C = A \otimes C + B \otimes C$. Next, the complex numbers a, and b commute with matrix A and B in multiplication, namely

$$\mathrm{a}\otimes\mathrm{A} = \mathrm{A}\otimes\mathrm{a} = \mathrm{aA} \tag{1}$$

$$\mathrm{aA}\otimes\mathrm{bB} = \mathrm{abA}\otimes\mathrm{B} \tag{2}$$

Then, the Kronecker product satisfies the distribution before multiplication, namely (A⊗B)(C$\otimes D) = AC\otimes BD$, when AC and BD are multiplicative. Even if they are not multiplicative, AB and C can be still multiplicative. Then the transposition of the product of the Kronecker product also satisfies the distribution rate, namely $(A \otimes B)^T = A^T \otimes B^T$. Next, we explain the definition of the operation symbol that appears below. $|A\rangle=\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ in C^n. For example, $|0\rangle=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $|1\rangle=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in C^2, and $|a\rangle$ as the linear combination of $|0\rangle$ and $|1\rangle$. We shall further refer to $|a, b\rangle = |a\rangle \otimes |b\rangle$ throughout the paper.

2.2. Schmidt decomposition and rank

In this subsection, we introduce the Schmidt decomposition of a bipartite state $|\varphi\rangle$ in $C^m \otimes C^n$. Suppose $|A_m\rangle$ is an orthonormal basis in C^m, and $|B_n\rangle$ is an orthonormal basis in C^n. We have $|\varphi\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{mn}\, |A_m\rangle\otimes|B_n\rangle$. Then the Schmidt decomposition of $|\varphi\rangle$is $\sum_{j=1}^{r}\sqrt{\gamma_j}\, |a_j\rangle\otimes|b_j\rangle$. Here the maximum r is less than or equal to the minimum value of m and n. Further, $|a_j\rangle$ and $|b_j\rangle$ are two sets of orthogonal normalized vectors of space any group of orthogonal normalized basis of space C^m *and* C^n, respectively. However, when we perform Schmidt decomposition (which is a restatement of singular value decomposition in a different context), we often want to know what the Schmidt rank is, which is difficult to determine directly by the computer. So we can perform the following conversion

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to determine the rank of the transformed matrix, which helps us get the result of Schmidt rank quickly. In the following, we introduce the way to decide the Schmidt rank by using unitary matrices **U** and **V**. Let $\begin{cases} \boldsymbol{U}|m\rangle = |a_m\rangle \\ \boldsymbol{V^T}|n\rangle = |b_n\rangle \end{cases}$. So we can rewrite the Schmidt decomposition of $|\varphi\rangle$ as follows.

$$|\varphi\rangle = \sum_{r=1}^{s} \sqrt{\gamma_r}\, |a_r\rangle \otimes |b_r\rangle \to |\varphi\rangle = \sum_{r=1}^{s} \sqrt{\gamma_r}\, \boldsymbol{U}|r\rangle \otimes \boldsymbol{V^T}|r\rangle \tag{3}$$

In this step, we transpose the part of the above formula $\boldsymbol{V^T}|r\rangle$ and change it into the $\langle r|\mathbf{V}$, where the upper and lower formulas can correspond one by one.

$$\sum_{r=1}^{s} \sqrt{\gamma_r}\, \boldsymbol{U}|r\rangle\langle r|\mathbf{V} \to \mathbf{U}\begin{bmatrix} \sqrt{\gamma_1} & \dots & 0 \\ \vdots & \sqrt{\gamma_2} & \vdots \\ 0 & \dots & \sqrt{\gamma_s} \end{bmatrix}\mathbf{V} \tag{4}$$

Since there is no linear relationship between these column vectors, we only need to know the rank of the above matrix to determine the Schmidt rank.

2.3. State Discrimination of the single system

Now, we propose a fundamental lemma for state discrimination. At the same time, we need to find a suitable construction method under the conditions so that we can judge the specific value of x through the output result (that is, the probability of the judge is 1, and the rest are 0. Here we give the first case.

Lemma 1. Suppose $|x\rangle \in \{|0\rangle, |1\rangle \dots |n-1\rangle\}$. Then we can determine $|x\rangle$ by quantum measurements.

Proof. The quantum measurement M_i's have to satisfy the measurement hypothesis. Mathematically, we have the conditions $\sum_{i=1}^{k} {M_i}^{+} M_i$=1 and $P(M_J,\, |x\rangle)$= $\langle x|{M_J}^{+}\ M_J|x\rangle$. The latter means that the probability of obtaining M_J. One can verify that $\sum_{i=1}^{k}\langle x|{M_i}^{+}\, M_i|x\rangle$=1, namely the probability sum is one. It means that one of the measurements M_i's must be realized. To construct M_i, we need to know the following fact: If $|a_1\rangle$…... $|a_n\rangle \in C^n$ are orthonormal, δ_{ij}=$\langle a_i|a_j\rangle$. Then, $\sum_{i=1}^{n} |a_i\rangle\,\langle a_i|$=$I_n$. To prove this fact, we assume V= $[|a_1\rangle\ \ \dots\ \ |a_n\rangle]^{n\times n}$ Then we have $V^{+} = \begin{bmatrix} \langle a_1| \\ \vdots \\ \langle a_n| \end{bmatrix}$. According to δ_{ij}=$\langle a_i|a_j\rangle$=0, δ_{ii}=$\langle a_i|a_i\rangle$=1. We could finally obtain

$$\mathrm{V}\ V^{+}=I_n=\sum_{i=1}^{n} |a_i\rangle\,\langle a_i| \tag{5}$$

Now, we return to the previous question: To compare M_J and find a way to confirm the value of x under the condition $\sum_{i=1}^{k} {M_i}^{+} M_i$=I Here we*note that* ($|x\rangle \in \{|0\rangle, |1\rangle \dots |n-1\rangle\}$). Cause in this first case that we gave you, there is this relationship: ${M_i}^{+}M_i = M_i = {M_i}^{+} = |i\rangle\langle i|$, so we have $M_i \in \{|0\rangle, |1\rangle \dots |n-1\rangle\}$. Then, we can verify that $\sum_{i=1}^{n-1} |n\rangle\,\langle n|=I_n$ Then, we link the whole process and output results with text illustrations below:

$$|x\rangle \begin{cases} |0\rangle\langle 0|x\rangle = \delta_{0x}|0\rangle, \quad P_0 = {\delta_{0x}}^2 \\ |1\rangle\langle 1|x\rangle = \delta_{1x}|1\rangle, \quad P_1 = {\delta_{1x}}^2 \\ \vdots \\ |n-1\rangle\langle n-1|x\rangle = \delta_{0x}|n-1\rangle, \quad P_{n-1} = {\delta_{n-1x}}^2 \end{cases} \tag{6}$$

At this point, we can find that the output probability of x is either 0 or 1. The output number with probability one is exactly x, in terms of quantum measurement theory. So, by using the above construction method, we have managed to complete our goal.

2.4. State Discrimination of Bipartite system

Now we intend to distinguish $|a\rangle$ and $|b\rangle$, where $|a\rangle$=$\frac{1}{\sqrt{2}}(|0,0\rangle+|1,1\rangle)$.$|b\rangle$=$\frac{1}{\sqrt{2}}(|1,0\rangle$+$|0,1\rangle)$are both in their Schmidt decomposition. We use the same constructor as before, which is $\sum_{i=1}^{n-1}|n\rangle\langle n|$=$I_n$ To multiply the two matrices, we multiply them by the identity matrix I_2 $below$. We write these two equations.

$$\begin{cases}(|1\rangle\langle 1|\otimes I_2)|x\rangle\\(|0\rangle\langle 0|\otimes I_2)|x\rangle\end{cases}\text{ where }|0\rangle\langle 0|+|1\rangle\langle 1|=I_2 \tag{7}$$

Then we plug x=$|a\rangle$and $|b\rangle$ into these two equations, respectively, and we end up with four results. Here we will explain the operation steps of one of the results, and the rest can be obtained by analogy. For (2) we plug in $|a\rangle$, then we get $\frac{1}{\sqrt{2}}(|0\rangle\langle 0|\otimes I_2))$ $|0\rangle\langle 0|$+$\frac{1}{\sqrt{2}}(|0\rangle\langle 0|\otimes I_2))$ $|1\rangle\langle 1|$ By using properties (A⊗B)(C⊗$D)=AC\otimes BD$ We finally get the result $\frac{1}{\sqrt{2}}|0,0\rangle$. The other three equations can be obtained by the same method, and finally, we have listed four results

$$\begin{cases}(|1\rangle\langle 1|\otimes I_2)|a\rangle=\frac{1}{\sqrt{2}}|1,1\rangle\\(|1\rangle\langle 1|\otimes I_2)|b\rangle=\frac{1}{\sqrt{2}}|1,0\rangle\\(|0\rangle\langle 0|\otimes I_2)|a\rangle=\frac{1}{\sqrt{2}}|0,0\rangle\\(|0\rangle\langle 0|\otimes I_2)|b\rangle=\frac{1}{\sqrt{2}}|0,1\rangle\end{cases} \tag{8}$$

Now we describe the process of the first step of the construction test. At this point, we find that in the first step, the probability can’t just be determined to be 0 or 1, but in the result, we determine one number, and the other number is uncertain. Therefore, the construction method and operation procedure of the first case needed to be used to carry out the second step of judgment, and we can reach the goal.

3. Non-orthogonal state discrimination of bipartite system

We found that in the above equations, we can distinguish the orthogonal normalized states, but can we distinguish the non-orthogonal states? Let us prove that the non-orthogonal states are indistinguishable by a proof.

We have shown that$\sum_{i=1}^{k}\langle x|M_i{}^{+}M_i|x\rangle$=I where $M_i{}^{+}M_i=E_i$. We assume that there are two non-orthogonal quantum states, $|\varphi_1\rangle$and$|\varphi_2\rangle$, and that they can be distinguished by quantum measurements. There must be the following equation $\langle\varphi_1|E_1|\varphi_1\rangle$=$I_1$, $\langle\varphi_2|E_2|\varphi_2\rangle$=$I_2$ $Since$ $\sum_{i=1}^{k}\langle x|M_i{}^{+}\mathrm{M_i}|x\rangle$=I,we can infer that $\langle\varphi_1|E_2|\varphi_1\rangle$=0, then, we set $|y\rangle$=$\sqrt{E_2}|\varphi_1\rangle$. Substituting into $\langle\varphi_1|E_2|\varphi_1\rangle$=0, we can find that $\sqrt{E_2}|\varphi_1\rangle=0$. Now let us decompose$|\varphi_2\rangle$ =a$|\varphi_1\rangle+b|\phi\rangle$, where $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are orthogonal. Thus available $|a|^2$+$|b|^2$=1, while $0<|b|<1$. Multiply both sides by $\sqrt{E_2}$,we get $\sqrt{E_2}|\varphi_2\rangle$=b$\sqrt{E_2}|\phi\rangle$. By defining a semidefinite matrix we can derive that $\langle\phi|E_2|\phi\rangle\leq\langle\phi|\phi\rangle$=1. From the above two conclusive equations, we can learn about the contradiction $\langle\varphi_2|E_2|\varphi_2\rangle=|b|^2\langle\phi|E_2|\phi\rangle\leq|b|^2$=1. So here we can conclude that non-orthogonal states cannot be distinguished by measurement.

Even increasing the auxiliary resources of product states $|a\rangle_{A1}\otimes|b\rangle_{B1}$ does nothing to help the quantum differentiation of non-orthogonal states. Descriptions are as follows. We now distinguish between two non-orthogonal vectors $|\alpha\rangle$ and $|\beta\rangle$, and add auxiliary entanglement quanta c and d to them, respectively, where $|\alpha\rangle$ and $|a\rangle$ are measured in the same group, so the subscripts are denoted by A or A1, and the same is true for $|\beta\rangle$ and $|b\rangle$. Then after adding the auxiliary entanglement quantum, the original a and b we want to distinguish become the following two formulas.

$$|\alpha\rangle_{AB}\otimes|a\rangle_{A1}\otimes|b\rangle_{B1} \tag{9}$$

$$|\beta\rangle_{AB}\otimes|a\rangle_{A1}\otimes|b\rangle_{B1} \tag{10}$$

Because$\langle\alpha|\beta\rangle \neq 0$, then by calculating, we have

$$(|\alpha\rangle_{AB}\otimes|a\rangle_{A1}\otimes|b\rangle_{B1})^{\mathrm{T}}|\beta\rangle_{AB}\otimes|a\rangle_{A1}\otimes|b\rangle_{B1} \neq 0. \tag{11}$$

Therefore, we conclude that even with the aid of quantum entanglement, it is still impossible to complete the measurement of non-orthogonal states.

4. State discrimination of orthogonal bipartite states assisted with product states

Then the conditions are more relaxed. If it has been determined that $|\alpha\rangle$ and $|\beta\rangle$ are orthogonal but difficult to measure, then whether the auxiliary resource of the product state can simplify the measurement and realize the measurement. Results are not achievable.

We have the following reasons. We can decompose $|\alpha\rangle$ and $|\beta\rangle$ as follows. It is decomposed into the (mn)-dimensional two-body space system

$$|\alpha\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}\,|i\rangle_A\otimes|j\rangle_B \tag{12}$$

$$|\beta\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}\,|i\rangle_A\otimes|j\rangle_B \tag{13}$$

$|i\rangle_A\ \epsilon C^m\otimes\ and\ |j\rangle_B \epsilon C^n$. Then, we combine the original formulas for easy and practical measurement.

$$|\alpha\rangle_{AB}\otimes|a\rangle_{A1}\otimes|b\rangle_{B1} \to \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}(|i\rangle_A\otimes|a\rangle_{A1})\otimes(|j\rangle_B\otimes|b\rangle_{B1}), \tag{14}$$

In addition, the state $|\beta\rangle$ is combined in the way $|\alpha\rangle\ does$. To simplify the next steps, we set $|i\rangle_A\otimes|a\rangle_{A1} = |x_i\rangle, |j\rangle_B\otimes|b\rangle_{B1} = |y_i\rangle$. Then we prove the existence of matrices U and V, where U$|x_i\rangle$=$|i\rangle$, V$|y_i\rangle$=$|j\rangle$. To prove U as an example, we can first prove the existence of A, that is A$|i\rangle$=$|x_i\rangle$. When $|i\rangle = |1\rangle$, $|2\rangle$..... $|m\rangle$, we can construct the matrix A=$[|x_1\rangle \quad |x_2\rangle \quad \ldots. \quad |x_m\rangle]$. Then, we have U=$A^{+}$. So far, we have proved the existence of U, and we can also prove the existence of matrix V. This is why we find that when U and V are applied to the original formula, that is, when written in the following expression, there is no change to the original $|\alpha\rangle\ and\ |\beta\rangle$.

$$(\mathrm{U}\otimes\mathrm{V})\,(\,|\alpha\rangle_{AB}\otimes|a\rangle_{A1}\otimes|b\rangle_{B1}) = |\alpha\rangle_{AB} \tag{15}$$

$$(\mathrm{U}\otimes\mathrm{V})\,(\,|\beta\rangle_{AB}\otimes|a\rangle_{A1}\otimes|b\rangle_{B1}) = |\beta\rangle_{AB} \tag{16}$$

So we can conclude that quantum entanglement assistance of product states does not bring any simple algorithm for measurement optimization.

5. State discrimination of bipartite system assisted with entangled states

In the previous discussion, we have been studying the distinction between two states, and this time we will study the distinction between three states.

Now let us assume that there are three pairwise orthogonal quantum states $|\alpha_1\rangle$,$|\alpha_2\rangle$, $|\alpha_3\rangle$. They are respectively $|00\rangle + |11\rangle, |00\rangle - |11\rangle$,$|01\rangle + |10\rangle$ When we apply the same quantum operation to each of them, they become the following three equations respectively, and to distinguish them further, they need to be orthogonal in pairs.

$$\begin{cases}(M\otimes\mathrm{I})(\,|00\rangle + |11\rangle)\\(M\otimes\mathrm{I})(\,|00\rangle - |11\rangle)\\(M\otimes\mathrm{I})(\,|01\rangle + |10\rangle)\end{cases} \tag{17}$$

These three equations need to be orthogonal in pairs. So we can write out three more equations and come to the following conclusion

$$\begin{cases}\langle 0|M^{\dagger}M|0\rangle - \langle 1|M^{\dagger}M|1\rangle = 0\\\langle 1|M^{\dagger}M|0\rangle - \langle 0|M^{\dagger}M|1\rangle = 0\\\langle 1|M^{\dagger}M|0\rangle + \langle 0|M^{\dagger}M|1\rangle = 0\end{cases} \tag{18}$$

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We already mentioned that M is a positive semi-definite matrix. Because here it is a two-by-two matrix, we can write it in the following form $\begin{bmatrix} a_{11} & b \\ b^* & a_{22} \end{bmatrix}$. By plugging this matrix into the three equations above, we get the following relationship a=c, b=0. This means that M can be written in the form of aI_2, This means that M is proportional to U and there is no practical significance in the step of distinguishing quantum states.

Now let us consider whether we can distinguish three quantum states using quantum assistance. Then the original three quantum states will become the following three forms

$$\begin{cases} (|00\rangle + |11\rangle)_{AB}(|00\rangle + |11\rangle)_{A_I B_I} \\ (|00\rangle - |11\rangle)_{AB}(|00\rangle + |11\rangle)_{A_I B_I} \\ (|01\rangle + |10\rangle)_{AB}(|00\rangle + |11\rangle)_{A_I B_I} \end{cases} \tag{19}$$

In actual operation, since the operator will operate the auxiliary resource and the original quantum state at the same time, we can equivalently convert the above formula into the following three formulas with practical operational significance.

$$\begin{cases} |00\rangle|00\rangle + |01\rangle|01\rangle + |10\rangle|10\rangle + |11\rangle|11\rangle \\ |00\rangle|00\rangle + |01\rangle|01\rangle - |10\rangle|10\rangle - |11\rangle|11\rangle \\ |00\rangle|00\rangle + |01\rangle|01\rangle + |10\rangle|10\rangle + |11\rangle|11\rangle \end{cases} \tag{20}$$

Let us use binary to simplify the above three formulas, We convert the three quantum states originally to be distinguished into the following three states.

$$\begin{cases} |\alpha_1\rangle = |00\rangle + |11\rangle + |22\rangle + |33\rangle \\ |\alpha_2\rangle = |00\rangle + |11\rangle - |22\rangle - |33\rangle \\ |\alpha_3\rangle = |02\rangle + |13\rangle + |20\rangle + |31\rangle \end{cases} \tag{21}$$

In the same way as above, we first construct a fourth-order semidefinite matrix, and since they are orthogonal in pairs, we can mathematically calculate the following property, where the lower angles are denoted by the elements determined by the rows and columns of the fourth-order semidefinite matrix. $a_{11}+a_{22}=a_{33}+a_{44}$, $a_{31}+a_{42}+a_{13}+a_{24}=0$, we also need to satisfy the following properties mentioned earlier, $\sum_{i=1}^{k} M_J^{+} M_J = I$. Secondly, in order to continue to distinguish, we have to verify whether the new quantum state is still pairwise orthogonal after the construction is complete. From these properties, we can conclude rank $M_J = 1$, which means it could be written in the following form, $M_J = |x_i\rangle\langle y_i|$. After construction and verification, we finally got the following four qualified M.

$$\begin{cases} M_1 = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}(0 \quad 1 \quad 1 \quad 0) \\ M_2 = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}(0 \quad 1 \quad -1 \quad 0) \\ M_3 = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}(1 \quad 0 \quad 0 \quad 1) \\ M_4 = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}(1 \quad 0 \quad 0 \quad -1) \end{cases} \tag{22}$$

One can verify that the resulting state $(M_j \otimes \mathrm{I})|\alpha_k\rangle$ is nonzero if and only if j=k. Hence, by obtaining a measurement result M_j, we can determine that the to-be-determined state is $|\alpha_j\rangle$. Below we give the calculation steps for the measurement result of M_1.

$$\begin{cases} (M_1 \otimes \mathrm{I})|\alpha_1\rangle = (|1\rangle + |2\rangle)(|1\rangle + |2\rangle) \\ (M_1 \otimes \mathrm{I})|\alpha_2\rangle = (|1\rangle + |2\rangle)(|1\rangle - |2\rangle) \\ (M_1 \otimes \mathrm{I})|\alpha_3\rangle = (|1\rangle + |2\rangle)(|3\rangle + |0\rangle) \end{cases} \tag{23}$$

Since M_1 is orthogonal to both $|1\rangle - |2\rangle$ and $|3\rangle + |0\rangle$, we can determine that the to-be-determined state is $|\alpha_1\rangle$.

6. Lower entanglement entropy completes the possibility of differentiation

First, the following formula definition of entanglement entropy is given here. Set the auxiliary entanglement resource to $|\phi\rangle$, So the Schmidt decomposition is as follows $|\varphi\rangle = \sum_{j=1}^{r} \sqrt{P_j}\, |a_j\rangle \otimes |b_j\rangle$. The expression for entanglement entropy is as follows,

$$\mathrm{E}(\varphi) = -\sum_{j=1}^{k} P_j \, log_2 P_j \tag{24}$$

It has some properties, such as the fact that its upper bound is $log_2 k$. This conclusion can be obtained by using the Chanson inequality. if and only if P_j is all equal. The inequality can be saturated. Its lower bound is 0, when the auxiliary resource is a product state, those with large entanglement entropy tend to consume more resources, while those with smaller entanglement entropy consume fewer resources. Moreover, from the previous papers, we found that when the entanglement entropy is 0, that is, the product state auxiliary resources cannot help quantum discrimination, and the maximum entangled state can be transformed into other entangled states, such as $(|00\rangle + |11\rangle)_{A_1B_1}$that we have proved before. So we can guess that the smaller the entanglement entropy is, the more difficult it is to distinguish quantum states. The following is our verification process for whether a smaller entanglement entropy can distinguish quantum states. Since the module length in a quantum-assisted resource is identical to 1, we assume that the auxiliary entanglement resource is $(\cos\alpha\, |00\rangle + \sin\alpha\, |11\rangle)_{A_1B_1}$

$$\begin{cases} (|00\rangle + |11\rangle)_{AB}(\cos\alpha\, |00\rangle + \sin\alpha\, |11\rangle)_{A_1B_1} \\ (|00\rangle - |11\rangle)_{AB}(\cos\alpha\, |00\rangle + \sin\alpha\, |11\rangle)_{A_1B_1} \\ (|01\rangle + |10\rangle)_{AB}(\cos\alpha\, |00\rangle + \sin\alpha\, |11\rangle)_{A_1B_1} \end{cases} \tag{25}$$

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$$\begin{cases} |\alpha_1\rangle = \cos\alpha\,|00\rangle + \sin\alpha\,|11\rangle + \cos\alpha\,|22\rangle + \sin\alpha\,|33\rangle \\ |\alpha_2\rangle = \cos\alpha\,|00\rangle + \sin\alpha\,|11\rangle - \cos\alpha\,|22\rangle - \sin\alpha\,|33\rangle \\ |\alpha_3\rangle = \cos\alpha\,|02\rangle + \sin\alpha\,|13\rangle + \cos\alpha\,|20\rangle + \sin\alpha\,|31\rangle \end{cases} \quad (26)$$

We still write the matrix $M^{\dagger}M$ and get the relationship as follows, we use e to represent the element in $M^{\dagger}M$,

$$\cos^2\alpha\, e_{11} + \sin^2\alpha\, e_{22} = \cos^2\alpha\, e_{33} + \sin^2\alpha\, e_{44}, \cos^2\alpha\, e_{31} + \sin^2\alpha\, e_{42} + \cos^2\alpha\, e_{13} + \sin^2\alpha\, e_{24} = 0 \quad (27)$$

One needs to similarly construct other equations using measurement operators and try to solve the equations later.

7. Conclusion

We have shown that product states do not help distinguish orthogonal states. We also have shown that three Bell states can be locally distinguished using a Bell state as an assisted quantum resource. The feasibility of using the most entangled state to distinguish quantum states from the feasibility of using smaller entangled states also proves the infeasibility of product states. An open problem arising from this paper is to distinguish more states in high dimensions assisted with entanglement, such as the set of nine two-qutrit maximally entangled states assisted with a two-qutrit entangled state. It is also an interesting problem to explore the possibility of implementing state discrimination by using two-way classical communications instead of the one-way counterpart used in this paper.

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