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On Diophantine equation $x^a \pm b = Dy^2$

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Abstract. Diophantine equation is an important part of the number theory and it has been widely studied for a long time. There are many studies on solving the integral solutions of Diophantine equations using algebraic methods. This paper uses documentation method, sums up the results of research in different documents of finding the integral solutions of some Diophantine equations in various conditions, especially the utilization of theories on Pell equation, which in the form of $x^a \pm b = Dy^2$. This paper mainly considers the situations when a =2 or a=3, which is a particular type of Diophantine equation. Several of the studies focus on the same equation but using different ways. Also, some subsequent studies on different equations are based on the former theorems provided by other writers and expand these theorems to broader applications. In order to emphasize the theorems obtained by the essays quoted and the methods the authors used, most of the mathematical procedures in their proofs are omitted.

Keywords: Diophantine Equation, Integral Solutions, Pell Equation, Algebra, Number Theory.

1. Introduction

Diophantine equation, also called indefinite equation, is named by the ancient Greek mathematician Diophantus. A Diophantine equation involving only sums, products, and powers in which all the constants are integers and the only solutions of interest are also integers [1]. The research on a Diophantine equation often base on three problems: 1. This equation has integral solutions in what conditions. 2. If the equation does have integral solutions, how many integral solutions are there? 3. Obtain the general solution. It is hard to determine whether a Diophantine equation has integral solutions or not and find the general solution without any specific method. The common methods are quadratic residue, congruence, Diophantine approximation, method of infinite descent and so on [2]. Pell equation is a kind of Diophantine equation, initially in the form of $x^2-Dy^2=1$, which is first studied by Fermat, who mentioned the equation $x^2-Dy^2=1$ again where D is a positive integer that is not a perfect square [3]. In the following paragraphs, this paper expands the theories of Pell equation and summarizes the theorems provided by some documents and the key points in their proofs about the integral solutions of the equations in the form of $x^a \pm b = Dy^2$where a=2 or a=3. The solutions of these equations not only promote the development of number theory, but also become the basis of a practical mathematical model that contribute to the application in many fields such as computer science and cryptography.

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2. On the equation $x^2 \pm b = Dy^2$

2.1. Theorems about $x^2+1=Dy^2$ and $x^2-4=Dy^2$

In the work by Chen in 2020[4], according to the application of Störmer theorem on Pell equations and the utilization of theories of Lehmer sequence, for the equation $x^2+1=Dy^2$, the relation between a group of positive integral solution (x,y) and its fundamental solution if there is specifically one or two prime divisors of y not dividing D is obtained, for the equation $x^2-4=Dy^2$, the relation between a group of positive integral solution (x,y) and its fundamental solution if there is exactly two prime divisors of y not dividing D is obtained. As a relevant theorem, the Störmer theorem states:

letting (x_1,y_1) be a group of positive integer solution of the equation $x^2-Dy^2=\pm1$, if all the prime divisors of y_1 divides D, then $x_1+y_1\sqrt{D}$ is the fundamental solution of the equation $x^2-Dy^2=\pm1$.

Chen mentioned and proved three theorems in this work, now we just consider one of them, which states:

letting (x,y) be a group of positive integer solution of the equation $x^2+1=Dy^2$, $y=p^n y'$, $p \nmid D$ is a prime number, D is not a perfect square, $n\in N$. If any prime divisor of y' divides D, then $x+y\sqrt{D}=\mathscr{E}$ or $\mathscr{E}q$, where q is an odd prime and $p \neq q$. $x+y\sqrt{D}=\mathscr{E}$ is the fundamental solution of the equation $x^2+1=Dy^2$[5].

In the proof of this theorem, Chen used the concept of congruence, Legendre symbol and quadratic residue. By Chen's theorems, all the equations consistent with the conditions can be determined as long as their fundamental solutions are found.

2.2. Theorems about $x^2-1=15y^2$ and $y^2-1=Dz^2$

Pu and Wan discussed the common solution of the equations $x^2-1=15y^2$ and $y^2-1=Dz^2$ in 2017[6]. Some relevant lemmas are:

Letting D>0 is not a perfect square, the fundamental solution of the Pell equation $x^4-Dy^2=1$ is (x0,y0), then the equation $x^4-Dy^2=1$ has a solution if and only if x_0 or $2x_0^2-1$ is a perfect square.

2. When a>0 is a perfect square, the equation $ax^4-by^2=1$ has at most one group of positive integer solution.

3. If D is a non-square positive integer, then the equation $x^2-Dy^4=1$ has at most one group of positive integer solution (x,y), also the equation has two groups of positive integer solution if and only if D=1785 or D=28560 or $2x_0$ and y_0 are both perfect squares, where (x_0,y_0) is the fundamental solution of the equation $x^2-Dy^2=1$.

4. (I) The only positive integer solution of the equation $x^4-15y^2=1$ is (x,y)=(2,1).

(II) The only positive integer solution of the equation $16x^4-15y^2=1$ is (x,y)=(1,1).

(III) The only positive integer solution of the equation $x^2-15y^4=1$ is (x,y)=(4,1).

5. Letting all the integral solutions of the equation $x^2-15y^2=1$ be (x_n,y_n), $n \in Z$, then

(I) x_n is a perfect square if and only if n=0 or n=1.

(II) $\frac{x_n}{4}$ is a perfect square if and only if n=1 or n=-1.

(III) yn is a perfect square if and only if n=0 or n=1.

The theorem provided by Pu and Wan suggests:

if $D=2p_1\ldots p_s$ $(1\le s\le3)$, $p_1,\ldots,p_s$ $(1\le s\le3)$ are distinct odd prime numbers, then the Pell equations $x^2-1=15y^2$ and $y^2-1=Dz^2$ have non-trivial solution (x,y,z)=(±244,±63,±8) and trivial solution (x,y,z)=(±4,±1,0) when D=2×33. When D=2×7×31×61, the equations have non-trivial solution (x,y,z)=(±15124,±3905,±24) and trivial solution (x,y,z)=(±4,±1,0).

To prove this theorem, letting (x,y,z)=(xn,yn,,z) where $n\in Z$ be the integer solution of the equations $x^2-1=15y^2$ and $y^2-1=Dz^2$. The conditions of different parities of n have to be considered respectively. The principally methods Pu and Wan used are considering some natures of odd and even numbers, recursive sequence and congruence.

2.3. Theorems about $x^2-1=6y^2$ and $y^2-4=Dz^2$

The study by Guan [7] gave two important theorems:

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If $p_1,\ldots,p_s$ are distinct odd prime numbers, $D=p_1\ldots p_s$ $(1\leq s\leq 3)$, then the simultaneous equations $x^2-1=6y^2$ and $y^2-4=Dz^2$ where $x,y,z\in Z$

have non-trivial solution $(x,y,z)=(\pm 49,\pm 20,\pm 6)$ and trivial solution $(x,y,z)=(\pm 5,\pm 2,0)$ when $D=11$

have non-trivial solution $(x,y,z)=(\pm 4801,\pm 1960,\pm 6)$ and trivial solution $(x,y,z)=(\pm 5,\pm 2,0)$ when $D=11\times 89\times 109$

have non-trivial solution $(x,y,z)=(\pm 4656965,\pm 1901198,\pm 840)$ and trivial solution $(x,y,z)=(\pm 5,\pm 2,0)$ when $D=11\times 97\times 4801$

only have trivial solution $(x,y,z)=(\pm 5,\pm 2,0)$ when $D\neq 11, 11\times 89\times 109, 11\times 97\times 4801$.

When D is even, if D has no prime divisor p which satisfies $p\equiv 1 \bmod 24$ and $p\equiv 7 \bmod 24$, then the simultaneous equations $x^2-1=6y^2$ and $y^2-4=Dz^2$ have the trivial solution $(x,y,z)=(\pm 5,\pm 2,0)$ only.

To prove the first theorem, letting $(\pm x_m,\pm y_m,,\pm z)$ be all of the solutions of the simultaneous equations $x^2-1=6y^2$ and $y^2-4=Dz^2$, where y_m and z must be even. There are two situations required to be discussed that $2\mid m$ or $2\nmid m$. To justify the two theorems, the writer mainly considered the parity of D in different conditions. What is more, in the demonstration of the second theorem, the writer converted the equations to $x^2-24a^2=1$ and $a^2-Db^2=1$ where $x,a,b\in N^*$. It is viable to do so since y and z can be expressed as $y=2a$, $z=2b$ where $a,b\in N^*$ and $D\in N^*$ has no square divisor and $D\equiv 2 \bmod 4$ when D is even.

3. On the equation $x^3\pm b=Dy^2$

3.1. Theorems about $x^3\pm 1=Dy^2$

For Diophantine equations $x^3-1=Dy^2$ and $x^3+1=Dy^2$ where $D>2$ and cannot be divided by 3 or other prime numbers in the form of $6k+1$. In 1942, Ljunggren attested that the equations $x^3-1=Dy^2$ and $x^3+1=Dy^2$ have at most one group of positive integer solution (x,y)[8]. In 1981[9], Ke and Sun demonstrated one theorem respectively for the equation $x^3-1=Dy^2$ and $x^3+1=Dy2$ that the only integral solution of the equation $x^3-1=Dy^2$ is $(x,y)=(1,0)$ and the only integral solution of the equation $x^3+1=Dy^2$ is $(x,y)=(-1,0)$, by using elementary methods only. For the equation $x^3-1=Dy^2$, first of all the authors utilized the method of factorization to express the equation as $(x-1)(x^2+x+1)=Dy^2$. It is obviously that the greatest common divisor of $(x-1)$ and (x^2+x+1) can be 1 or 3 only, then the two situations that $\gcd(x-1,x^2+x+1)=1$ or $\gcd(x-1,x^2+x+1)=3$ were considered severally. After a few steps, the situation that $\gcd(x-1,x^2+x+1)=1$ was shown to be impossible. Therefore, as $\gcd(x-1,x^2+x+1)=3$, $x-1$, x^2+x+1 and y can be expressed in the form of u and v $(u>0,\ v>0)$ that $x-1=3Du^2$, $x^2+x+1=3v^2$, $y=3uv$. In the following manipulations, $x=3Du^2+1$ was substituted to $x^2+x+1=3v^2$, a new equation was then obtained. Two conditions whether D is divisible by 2 also have to be discussed respectively since it is related to two different expressions of u. The highlight of the study by Ke and Sun is that they used elementary methods such as factorization and quadratic residue only.

3.2. Theorems about $x^3\pm 27=Dy^2$

In 1988, in order to add the work by Ke and Sun in 1981[9], Cao provided a new theorem severally for the equation $x^3-1=3y^2$ and $x^3\pm 1=6y^2$ that the only integral solution for $x^3-1=3y^2$ is the trivial solution $(x,y)=(1,0)$ and the only integral solution for $x^3\pm 1=6y^2$ is the trivial solution $(x,y)=(\pm 1,0)$[10]. Cao also used the method of factorization to turn $x^3-1=3y^2$ into $(x-1)(x^2+x+1)=3y^2$, then expressed $x-1$, x^2+x+1 and y as $x-1=9a^2$, $x^2+x+1=3b^2$, $y=3ab$, substituted x to x^2+x+1 for the following manipulations. Before the theorem about $x^3\pm 27=Dy^2$, Cao stated two prerequisite theorems:

For the equation $x^3+1=3Dy^2$ where $D>0$, $D=1$ or D has no square divisor and cannot be divisible by the prime numbers in the form of $6k+1$, the only integral solution of the equation $x^3+1=3Dy^2$ is the trivial solution $(x,y)=(-1,0)$.

For the equation $x^3-1=Dy^2$ where $D>0$, $D=1$ or D has no square divisor and cannot be divisible by the prime numbers in the form of $6k+1$, the only integral solution of the equation $x^3-1=Dy^2$ is the trivial solution $(x,y)=(1,0)$.

By extending these two theorems, Chen testified that for the equation $x^3+27=Dy^2$ where D>0 and D has no prime odd power divisor in the form of 6k+1, except for some values of D that the equation does have at most two non-trivial integral solutions, the equation has trivial solution (x,y)=(-3,0) only. To prove this theorem, the lemmas that the equation $x^3+1=y^2$ has integral solutions (0,±1) and (2,±3) only besides the trivial solution (-1,0) and the equation $x^3+1=2y^2$ has integral solutions (1,±1) and (23,±78) only besides the trivial solution (-1,0) are needed. Another theorem provided by Cao about the equation $x^3-27=Dy^2$ suggests that for the equation $x^3-27=Dy^2$ where D>0 and D has no prime odd power divisor in the form of 6k+1, except for the situation when D=2 the equation has solution (5,±7) only and the situation when D=98 the equation has solution (5,±1) only, the only solution of the equation $x^3-27=Dy^2$ is the trivial solution (3,0). To prove this theorem, the two conditions $3 \mid x$ or $3 \nmid x$ have to be considered.

3.3. *Another way to solve the equation $x^3 \pm 27=Dy^2$ and the theorems about $x^3 \pm 729=Dy^2$*

Founded on the study on the equation $x^3 \pm 1=Dy^2$by Ke in 1988[9], Si and Pang provided all the non-trivial solutions of the equation $x^3 \pm 27=Dy^2$(D>0, has no divisor of perfect square and prime numbers in the form of 6k+1) in another way and all the non-trivial solutions of the equation $x^3 \pm 729=Dy^2$[11]. Lemmas:

1. For the equation $x^3+1=Dy^2$,

(I) when D is not the multiple of 2 or 3, the integral solutions are (-1,0), (0,1), (2,3).

(II) when D is a multiple of 2, the integral solutions are (-1,0), (0,1), (23,78).

(III) when D is a multiple of 3, the integral solution is (-1,0).

2. For the equation $x^3-1=Dy2$, the only integral solution is (1,0).

In theorems by Si and Pang, the non-trivial solutions of the equation $x^3+27=Dy^2$when D=3, 6, 11, the only non-trivial solution of the equation $x^3-27=Dy^2$when D=2, the non-trivial solutions of the equation $x^3+729=Dy^2$when D=33, 1, 2, 74, the non-trivial solutions of the equation $x^3-729=Dy^2$when D=47, 6 are given. To prove the theorems above, the situations of $3 \mid x$ or $3 \nmid x$ have to be considered.

4. Conclusion

This paper discussed mainly on the Diophantine equation $x^a \pm b=Dy^2$where a=2 or a=3. All of the theorems above consummate our study on the Diophantine equation. However, the researches on the Diophantine equations are so broad that the content of this paper is only the tip of the iceberg. Even one particular equation can be solved in many different ways, the methods mentioned in this paper can also be used to solve other equations. The research orientation in the future is to use known methods to solve equations of other values of b and D, also the discovery of new methods.

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