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A brief analysis of mathematical thinking and problem-solving ideas in advanced mathematics

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Abstract. In this paper, some points were summarized to help students improve their advanced mathematical thinking after analyzing various problem-solving ideas and many advanced mathematical thinking points. The sources for the review are almost papers that were published from 1994 to 2022. This time period was chosen because advanced mathematics has developed greatly during this period, and many excellent papers have emerged, which can be better to refer to and draw conclusions. The review of this paper is guided by major mathematical journals. The review summarizes the research themes, research objectives, research methods, and conclusions reached in different papers. The paper also finds many kinds of methods to solve advanced mathematic problems. At the end of the paper, probabilistic thinking, infinite thinking, functional thinking and calculus thinking five kinds of thinking were proposed to help people to learn about the mathematical thinking in advanced mathematics.

Keywords: Advanced Mathematical Thinking, Probabilistic Thinking, Infinite Thinking, Functional Thinking, Calculus Thinking.

1. Introduction

At present, many countries pay more and more attention to mathematics education, and the direction of mathematics education has changed from traditional teaching to guiding students to improve their mathematical thinking and form their own mathematical thinking system through learning many different kinds of problem-solving ways. This makes the problem-solving method extremely important, because the problem-solving method reveals the author's way of thinking [1]. This paper discusses how to improve mathematical thinking and various problem-solving methods, and mainly summarizes the papers published in major domestic and foreign journals from 1994 to 2022. The author wanted to select papers from 2000 to 2022, but considering the rapid development of mathematics at the end of the 20th century, she finally selected papers from 1994 to 2022 and classified them.

In this review, it is mainly divided into the following parts. First of all, what is advanced mathematical thinking and the importance of advanced mathematical thinking are defined. Then it describes the selection method of our guiding documents and the points that need attention. Afterwards, for the topic, the results are presented:

What are the commonalities and differences between the methods and theories mentioned in the paper results from 1994 to 2022.

What does the paper contribute to the research and what are the future prospects of the research.

Some of the author's own views on this topic.

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2. Meaning of advanced mathematical thinking

Advanced mathematical thinking is a very abstract concept that is difficult to describe. It is difficult for people to define it with a simple sentence. Usually, advanced mathematical thinking refers to a way of thinking and ability required to solve more complex, abstract and in-depth mathematical problems. This type of thinking involves a deep understanding of mathematical concepts, abstract thinking, logical reasoning, problem modeling, creative thinking, and the flexible application of mathematical skills. Advanced mathematical thinking usually begins to develop at the undergraduate level, covering areas such as calculus, linear algebra, differential equations, and complex variable functions. And according to the research, the advanced mathematical thinking is divided into three stages: entry, attack and review. The author puts forward the idea of AMT (an essential part in solving mathematical problems), some of the ideas contained in AMT will eventually form a part of advanced mathematics thinking. Finally, David Tall puts forward five characteristics of advanced mathematics thinking [2]. Although the final conclusion of this paper is that it is still difficult to define advanced mathematical thinking, it provides people with a new way of thinking. About the process of the advanced mathematical thinking, many researchers have done a lot of research. They think mathematical thinking is associated with psychology and the difference between low-level thinking and high-level thinking is that the complexity of understanding is different, and the corresponding psychological qualities are also different [3]. Different studies have different views on advanced mathematical thinking, so it is difficult to define advanced mathematical thinking, but it is not difficult to see that advanced mathematical thinking is important in all aspects of life.

3. Why advanced mathematical thinking is so important

There are many reasons to say that the advanced mathematical thinking is very important, but the best reason is its function in the problem-solving aspect. Firstly, it can improve our critical thinking, enhance students' logical reasoning ability. Besides this, advanced mathematical thinking can help cultivate abstract thinking, understand abstract concepts and apply them in advanced mathematical problems, thereby better understanding the nature of the advanced mathematical problems which often involve abstract concepts and symbols. According to a research, the author Chairul Anami, Budi Usodo, Sri Subanti and their colleagues did an experiment, there are 29 student from a class, the 29 students have different mathematical abilities and mathematical thinking. After evaluating them, the students with high mathematical ability got level of C6, it means they have the ability to create new things, while the students with low mathematical ability can only reach the level of C4, it represented that these students only have the ability to analyze problems in mathematics, but do not have the ability to solve problems [4]. It can be seen from this experiment that mathematical thinking represents mathematical ability to a certain extent, that is, the ability to solve problems. Therefore, mathematical thinking is extremely important both in mathematics and in other aspects of people's lives.

4. The different ways of problem-solving

4.1. Probability theory

The first common ways to solve the mathematical problem is probability theory. Probability theory appeared in the 17th century, and people have been trying to understand probability theory. Nowadays, research results on probability theory are becoming increasingly mature. For the definition of probability theory, it can be expressed as assuming that there are N events in a whole, assuming if each event has the same probability of occurrence, the event M consists of n event groups, then the probability of the event happening is n/N. As for its application in higher mathematics problems, probability theory can simplify some abstract problems in higher mathematics and make them easier to understand [5]. There is an example from one research:

Problem: solve $\sum_{a=1}^{\infty} a^2 (\frac{2}{3})^{a-1}$

Solution: Assume $P = \frac{1}{3}$, ε is a random variable, then

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$$P(\varepsilon = a) = \frac{1}{3}\left(\frac{2}{3}\right)^{a-1}, E(\varepsilon) = 3, D(\varepsilon) = 6$$

It is obvious to see that $E\varepsilon^2 = E(\varepsilon)^2 + D(\varepsilon) = 15$

$$E\varepsilon^2 = \lim_{a=1} a^2 \frac{1}{3} (\frac{2}{3})^{a-1} = \frac{1}{3} \lim_{a=1} a^2 \frac{1}{3} (\frac{2}{3})^{a-1} = 15 \quad (1)$$

So it is easy to get the answer, which is 45.

Problems of series type like this can usually introduce variance and expectation, which can simplify the problem and make it easier to solve [6]. Another example is below:

Problem: solve $\int_{-\infty}^{+\infty} e^{\frac{-(x-u)^2}{2a^2}} dx$

Solution: According to the idea of probability, replace the content of different elements of the definite integral, and get $X \sim N(u, \sigma^2)$,

At this time, the probability density function can be designed as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-u)^2}{2a^2}}, x \in R$$

According to the principle of uniformity $\int_{-\infty}^{+\infty} f(x)dx = 1$, we can get

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-u)^2}{2a^2}} dx = 1,$$

$$\int_{-\infty}^{+\infty} e^{\frac{-(x-u)^2}{2a^2}} dx = 2\pi\sigma.$$

It can be seen that the definite integral after simplification can be transformed with the idea of probability, and it becomes very easy to solve. It can be seen from the above two examples that probability theory plays a vital role in solving advanced mathematics problems, and appears as a common solution in advanced mathematics.

4.2. Multiple answers

In advanced mathematics problems, multiple solutions often appear [7]. The following example is the most common one.

Problem: Assume the equation $xy^2 + e^y = \cos(x + y^2)$, solve y'.

Solution1: Deriving both sides of the equation with respect to x.

$$y^2 + 2xyy' + e^y y' = -\sin(x + y^2)\,(1 + 2yy') \quad (2)$$

$$y' = -\frac{y^2 + \sin(x + y^2)}{2xy + e^y + 2y\sin(x + y^2)}.$$

Solution2: Let $F(x, y) = xy^2 + e^y - \cos(x + y^2)$

$$F'_x = y^2 + \sin(x + y^2)$$

$$F'_y = 2xy + e^y + 2y\sin(x + y^2)$$

$$\frac{dy}{dx} = \frac{y^2 + \sin(x + y^2)}{2xy + e^y + 2y\sin(x + y^2)}. \quad (3)$$

Solution3: $d(xy^2 + e^y) = d(\cos(x + y^2))$

$$y^2dx + 2xydy + e^ydy = -\sin(x + y^2)(dx + 2ydy)$$

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$$[(2xy + e^y) + 2y\sin(x + y^2)]dy = -[y^2 + \sin(x + y^2)]dx$$

$$\frac{dy}{dx} = -\frac{y^2 + \sin\left(x + y^2\right)}{2xy + e^y + 2y\sin(x + y^2)}. \tag{4}$$

4.3. Reverse thinking

Reverse thinking is also widely used in advanced mathematics problem solving. Reverse thinking can expand students' problem-solving ideas, cultivate students' innovative thinking, and enable students to solve difficult advanced mathematics problems skilfully. Reverse thinking mainly includes proof by contradiction and backward reasoning, the principles of which are to think about problems backwards [8]. The following will demonstrate the application of proof by contradiction and backward reasoning through two examples:

Problem1: Let $x, y \in N$, such that $x \neq y$ and $x, y > 0$. Prove that $xz^2 + yz + (y - x) = 0$, has no positive integer root z.

Proof: Assume that z is a positive integer root.

So

$$z = \frac{-y \pm \sqrt{y^2 - 4x(y - x)}}{2x} = \frac{-y \pm (2x - y)}{2x} \tag{5}$$

We can get $z_1 = \frac{-y+2x-y}{2x} = 1 - \frac{y}{x} < 1$

$$z_2 = \frac{-y - 2x + y}{2x} = -1 < 0$$

It is a contradiction, so the equation has no positive integer root z.

The above problem-solving process is the process of proof by contradiction. First, assuming that the proposition is correct, and after a series of calculations, a wrong answer is obtained, then the proposition can be proved to be wrong. Here is an example of an application of backward reasoning:

Problem: Let f(x) be continuous on $[a, b]$, second-order derivable in (a, b),

the line segment connecting $A(a, f(a))$ and $B(b, f(b))$ intersects the curve $y = f(x)$

For $C\left(c, f(c)\right)(a < c < b)$, prove that there is at least one point δ in (a, b) such that

$$f''(\delta) = 0.$$

Proof: From the perspective of backward reasoning, we should consider using Rolle's theorem for the first-order derivative function, because $A\left(a, f(a)\right), B\left(b, f(b)\right), C\left(c, f(c)\right)$ are the same line, we can get

$$\frac{f(c) - f(a)}{c - a} = \frac{f(b) - f(c)}{b - c} = \frac{f(b) - f(a)}{b - a} = K_{AB}(K_{AB} \text{ is the slope of AB})$$

And because $f(x)$ satisfies the conditions of Lagrangian theorem on $[a, c]$ and $[c, b]$, then

$$f'(\delta_1) = \frac{f(c) - f(a)}{c - a}, (a < \delta_1 < c)$$

$$f'(\delta_2) = \frac{f(b) - f(c)}{b - c}, (c < \delta_2 < b)$$

Then use Rolle's theorem for $f'(\delta)$ on $[\delta_1, \delta_2]$:

$$f''(\delta) = 0, \delta \in (\delta_1, \delta_2) \subset (a, b). \tag{6}$$

This question uses the backward reasoning method, using known conclusions to deduce the possible reasons for the conclusions, and get the results step by step. The method of proof by contradiction and inversion is often used in proof questions, because proof questions usually give the final conclusion, and

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they are not commonly used in calculation questions. But using this method for a long time will broaden our thinking and improve our mathematical thinking.

4.4. Basic number knowledge

It is not a direct method to solve problems, but it can improve the ability of mathematical inductive reasoning, and mathematical inductive reasoning is a very common way to solve problems, so it can show that basic numerical knowledge can improve the mathematical thinking ability, and then better solve problems. For the view that the basic numerical knowledge can improve the ability of mathematical inductive reasoning, some people have also done the relevant experiment. Lisa A. Haverty conducted an experiment with seventh grade students. The professor taught the seventh-grade students a set of basic number knowledge. After the training of number knowledge for the students, they did a test for the students about their mathematical inductive reasoning ability. At the same time, Lisa A. Haverty developed a cognitive model of adult inductive reasoning. Both the model and the test showed that numerical knowledge had a positive correlation with mathematical inductive reasoning. This experiment also highlights the importance of basic numerical knowledge in solving advanced mathematical problems. So it can be seen from this experiment that when the students are learning advanced mathematics, while mastering problem-solving methods, they also need master the basic knowledge of numbers, which can help them better improve their problem-solving ability, and expand their thinking and improve their mathematical thinking ability [9].

4.5. Mathematical literacy skills

Mathematical literacy skills plays an important role in the problem analysis of advanced mathematics, and it is related to students' understanding of the topic. Many scholars are also very interested in this kind of skills. Regarding this issue, they conducted an experiment, they tested several seventh-grade students in eight aspects and used different models to analyze the survey results. Students are graded based on the scores they receive. Survey results show that mathematical literacy skills can improve students' ability to think about simple mathematical problems. Although this experiment was conducted on middle school students, it is still representative. It can be seen that mathematical literacy skills are important in improving advanced mathematical thinking [10].

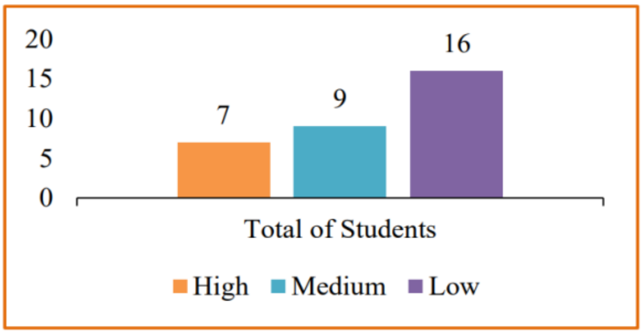

Figure 1. Results of Mathematical Skills Test [9].

5. Conclusions

The five ways of solving problems are summarized above, in order to find different advanced mathematical thinking in the process of solving problems. After analyzing the above problem-solving methods, the purpose of learning mathematics needs to be talked about. When people learn mathematics, they do not simply learn how to solve problems, but learn to use tools to solve some real-life problems; learning mathematics requires learning mathematical thinking and applying mathematical thinking to life as a way of thinking; at the same time , learning mathematics can also improve their self-awareness. Mathematics should be controlled by people, so that the more mathematics they learn, the deeper their understanding of trends will be. These are the real reasons why people study mathematics.

Finally, based on the above five problem-solving methods and scholars' analysis of advanced mathematical thinking [11], several main ways of thinking in higher mathematics have been summarized. The thinking to be talked about is not purely mathematical thinking, the thinking people should learn should be thinking that is helpful to the world.

(1) Probabilistic Thinking: Many things in the world are not certain and immutable. When people learn to look at the world with the perspective of probability, they will become less blind. This is the first important thinking in advanced mathematics.

(2) Infinite Thinking: The world is dynamic, not static. Everything in the world is in motion all the time, which reflects a trend, a trend of infinite increase. Therefore, people need to master the idea of infinity, and then better grasp the laws of world change. This is the second major thinking in advanced mathematics.

(3) Functional Thinking: As the name suggests, functional thinking means that one variable changes as another variable changes. In life, it is reflected in the fact that one thing can change as another thing changes. Therefore, things are interconnected, which helps them understand things better. This is the third thinking in advanced mathematics.

(4) Calculus Thinking: Calculus thinking refers to microscopic calculations. This kind of thinking allows people to look at the world from a dynamic perspective. It requires not only starting from the subtleties, paying attention to observation, but also needing to grasp the macroscopic laws of things. This is the fourth major thinking in advanced mathematics.

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