The International Conference on Computing Innovation and Applied Physics (CONF-CIAP 2022)
DOI: 10.54254/2753-8818/2/20220084

Some Fundamental Results from Complex Analysis

Xuhan Pan*

Beijing Aidi School, Beijing, China

a1707201@aidi.edu.cn

Abstract. This paper is going to introduce the most basic theory of analytic functions of one complex variable. It begins at the discussion of meaning of complex number and the historical development from the formula of cubic equation to the square root of negative number. In the middle section, which is divided in to four small parts. First part states the expression of complex number z, algebraic properties, and the relationship of each single complex number with whole complex plane. Second part concerns about several elementary functions of complex number. Next part relates to the derivative of complex number, such as the partial derivative and Cauchy-Riemann Equation to verdict that whether a function is analytic in the domain. Last part is the integral of complex functions, which concludes some important theorems with the proof, and examples. After the section of introduction of complex number, it comes to the final section which is talking about an identity involving complex number and the prove of it.

Keywords: Complex Analysis, Complex Variable

1. Introduction

Complex numbers are useful, even though most people don't generally recognize their utility in real life. For complex number, there is a strong evidence to show the usefulness of complex number, which is the Fundamental Theorem of Algebra. This theorem states an idea that a univariate polynomial with nth degree whose coefficients are entailed all real numbers has and only has n complex roots. That is, complex numbers give solutions to all polynomial functions that satisfy combine with real coefficient. For example, a quadratic function with an upward opening and positive constant term has no intersection with the x-axis. However, this does not mean that the equation has no solution. Rather, the equation has no real solution, but a complex solution. We cannot observe directly these complex solutions geometrically, we can only imagine that the equation has two solutions-the imaginary solutions. Therefore, complex solutions need a higher latitude to represent their existence. In this kind of situation, we need to use the complex plane. In the plane, imaginary numbers are displayed on the complex plane, and they represent as a vector who has coordinate combining with its real and imaginary part. A complex number also represent as a quantity of rotation. Any real number multiplied with i that is equivalent to geometrically rotating this real number by 90° counterclockwise; and multiplying it with two i is equal to rotate 180° to obtain a number that is only symbolically opposite to this real number. In geometry, we can clearly see that the real numbers R and $-R$ are on the same real axis, with a difference in angle of 180°. Meanwhile, complex number is also applied in other subject that are more practical to people. For instance, in electronics, where complex numbers can be used to represent the status of circuit elements. And it uses in quantum mechanism, which base on an infinitely-dimensional Hilbert space on a complex number field. Since complex numbers are used in so many disciplines, lets us discuss some properties and computations of complex numbers in this paper to help people understand some basic information about complex number.

2. The History of Complex Number

Imaginary number originated from an article written by G.Cardano, which is how to make $ab = 40$ when $a + b = 10$? There is the answer $(5 + \sqrt{-15})(5 - \sqrt{-15}) = 40$. But it is difficult for people to understand this answer, because people thought that there was no meaning of square root of negative number during that period. Of course, in modern times, people are still affirming the role of complex numbers. As for understanding why people change their minds about complex number, we have to clear its history. Let's start by solving the equations.

The solution of the quadratic equation originated in the ninth century AD with the formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. With this equation, we notice that when $\Delta = \sqrt{b^2 - 4ac} \leq 0$, the equation has no solution. By the 14th century, however, people were keen to solve cubic equations. Scholar Fiore challenged mathematician Tartagila to solve cubic equations, later one who succeeded in deriving a general solution to cubic equations and thus, gained fame[1-10]. Cardano knew of this thing, consulted the equation from Tartagila and re-proof the formula, which had been evolved to the famous Cardan formula $x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$ [1-10]. Cardano noticed that the formula needed to square root to the negative numbers, even though the result they ended up with was a real number[1-10]. At this time, people began to consciously introduce new algebraic operations which is the complex number. Bombelli R. used Cardano's formula to calculate the roots cubic equation and obtained the fact that something equals to the square root of -1[1-10]. In the 17th century, Descartes first introduced the Cartesian coordinate system and used the name imaginary number to describe the square root of -1[1-10]. In 1777, Euler used i in his paper to represent imaginary units. Until 1799, C.Wessel first published a geometric interpretation of imaginary numbers[1-10]. Like the real numbers can be represented by a number axis, that is, a straight line with an origin point, positive numbers lying on the right axis of axis, and negative lying on other side. Wessel realized that complex numbers can be represented as vectors on the plane, with the abscissa as the real part and the ordinate as the imaginary part. Thus, complex number has been built lots of properties. Overall, all of these lead to the recognition of imaginary numbers by people.

3. The Numeric Properties of Complex Number

3.1. The Expression of Complex Number

3.1.1. What is Complex Number. In the beginning, people abstracted numbers from nature to count the actual matters, and called them natural numbers. However, as more calculation been created, people expand larger number systems. Complex number is the expansion of real number when there are some problems cannot be settled in real number, such as the square root of negative number. Imaginary number i which is the unit of complex number is defined as $\sqrt{-1}$. And complex is represented as $z = x + yi$, which consists real part x and imaginary part yi represented by $x = Re(z)$ and $y = Im(z)$. A complex number which has zero real part is called $purely\ imaginary$. Complex number is showed on the complex plane as a point. The set of all complex number is defined as C. For example, i is showed on point $(0,1)$, which has coordinate (x, y) relating to their real and imaginary part.

3.1.2. Calculation of Complex Number. $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$$z_1z_2 = (x_1x_2 - y_1\,y_2) + i(x_1x_2 + y_1\,y_2).$$

Commutativity: $z_1 + z_2 = z_2 + z_1$ and $z_1z_2 = z_2z_1$ for all $z_1, z_2 \in C$.

DOI: 10.54254/2753-8818/2/20220084

Associativity: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$; and $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ for all $z_1, z_2, z_3 \in C$.
Distributivity: $(z_1 + z_2) z_3 = z_1 (z_2 + z_3)$ for all $z_1, z_2, z_3 \in C$ [6].

The modulus written as $|z| = \sqrt{x^2 + y^2}$ of complex number z means that the length between the number and original point.

When complex number has its meaning, people define the algebraic number and transcendental number. A univariate polynomial, where the coefficients of it are all integers, writes like

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0 (n\ belongs\ positive\ integer) \tag{1}$$

The root of this equation is called an algebraic number, otherwise it is called a transcendental number. Some common algebraic numbers include all rational numbers, Gaussian integers, etcetera; e and π are both transcendental numbers.

The argument: argument is defined an angle that demonstrate the angle between complex number z and the positive real line. This angle is equal to $arg(z) = Arg(z) + 2\pi$. Where Arg is the principal angle of the complex number, which is between $[-\pi, \pi]$.

The exponential expression and trigonal expression of complex number

$$\boldsymbol{z = x + yi = r(cos\theta + isin\theta) = re^{i\theta}} \tag{2}$$

(r is the modulus of complex number $\boldsymbol{z}$ and $\boldsymbol{\theta\ equal\ to\ the\ Arg(z)}$)

The conjugate number of $z = x + yi$ is $\overline{z} = x - yi$. Conjugate complex number has two properties: first one, the modulus of z equal $\overline{z}$, $|z|$=$|\overline{z}|$. And according to Vieta's Theorem, people find that in the root polynomial, a complex number z always appears with $\overline{z}$. Square root of -1 equal to i and $-i$. The creation of conjugate number is to perfect this system-the paired roots of complex number of polynomials. We can denote the real and imaginary parts by conjugate paired number: $Re(z) = \frac{z + \overline{z}}{2}$ and $Im(z) = \frac{z - \overline{z}}{2i}$

The powers of complex number, the De Moivre formula concludes that

$$z^n = r^n e^{in\theta} = r^n (cosn\theta + isinn\theta) \tag{3}$$

Triangle Inequality and other inequality:
For all $z \in C$, $|Re(z)| \leq |z|$ and $|Im(z)| \leq |z|$, and for all $z, w \in C$ $||z| - |w|| \leq |z - w|$ [5]

$$\left||a| - |b|\right| \lhd |a - b|; \quad |a - b| \lhd |a| + |b|$$

3.2. *Elementary Function of Complex Number*

3.2.1. Parameters of Complex Function. Let G be the set of complex numbers $z = x+iy$. There is a specific rule that makes each element in the set G having one or more complex number w corresponding to it. Then the complex variable w is a function of the complex variable z, that is, the complex function. Complex function represents as $w = f(z) = u(x, y) + iv(x, y)$. The independent variable is $z = x + iy$, responsible variable $w = u + iv$. If each z is corresponding to single w: $z_1 \neq z_2$, having point $f(z_1) \neq f(z_2)$, we call it single-valued complex function. Otherwise, it is many-valued complex function.

Polar form of complex function, that is, transfer variable z in the exponential form $re^{i\theta}$. At this time, $f(z) = u(x, y) + iv(x, y)$ should be switched to $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$.

3.2.2. Mapping of function. The mapping of a complex variable function is equivalent to the mapping of two binary real functions $u = u(x, y), v = v(x, y)$ [10].

For example, a function $z = re^{i\theta}, w = f(z) = z + \frac{1}{z}$.

$$u(x, y) = rcos\theta + r^{-1}cos\theta = \left(r + \frac{1}{r}\right)cos\theta, v(x, y) = rsin\theta - r^{-1}sin\theta = (r - \frac{1}{r})sin\theta \tag{4}$$

From simple transform, we can notice that:

$$cos\theta = \frac{u(x,y)}{r + \frac{1}{r}}, \text{and } sin\theta = \frac{v(x,y)}{r - \frac{1}{r}} \tag{5}$$

According to $sin^2\theta + cos^2\theta = 1$, we obtain an ellipse function

$$\left(\frac{u}{r+\frac{1}{r}}\right)^2 + \left(\frac{v}{r-\frac{1}{r}}\right)^2 = 1 \tag{6}$$

3.2.3. Basic Forms of Elementary Function of Complex Number. Logarithmic function and exponential function:

$$Log[z] = Log[(re)^{i\theta+2k\pi}] = Log[r] + Log[e^{i\theta+2k\pi}] = Log[r] + iarg(z) \tag{7}$$

$$e^z = e^x\,(cosy + isinx) \tag{8}$$

Trigonometric function:

$$cosz = \frac{e^{iz} + e^{-iz}}{2} sinz = \frac{e^{iz} - e^{-iz}}{2} \tag{9}$$

Power function:

$$z^c = e^{clogz} \qquad e^{logz} = z \qquad Log[e\text{^}z] \neq z \tag{10}$$

3.3. Derivative of Complex Number

3.3.1. Limits. When function is continues at point z_0, and

$$\lim_{x\to x_0} f(z) = \lim_{y\to y_0} f(z) = w_0 \tag{11}$$

The limit of f at z_0:

$$\lim_{z\to z_0} f(z) = w_0 \text{ [3]} \tag{12}$$

3.3.2. Derivative. Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \varepsilon$ of a point z_0. The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z- z_0}, \tag{13}$$

The function is said to be differentiable at z_0 when $f'(z_0)$ exists.

Cauchy-Riemann Equation [3] To determine whether $f'(z_0)$ exist, we should use the Cauchy-Riemann Equation, or the C-R Equation

$$u_x = v_y, u_y = -v_x. \tag{14}$$

If function satisfied this equation, it would differentiate.

Proof: Complex number z_0, the limit of z_0 direct toward z_0 horizontally which only changes x part of function. Value of this limit equal to

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{u(x, y_0) - u(x_0, y_0) + i(v(x, y_0) - v(x_0, y_0))}{x - x_0} = u_x(x_0, y_0) + iv_x(x_0, y_0) \tag{15}$$

And limit of z_0 direct toward z_0 vertically which only changes y part of function. Value of this limit equal to

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{u(x_0, y) - u(x_0, y_0) + i(v(x_0, y) - v(x_0, y_0))}{i(y - y_0)}$$
$$= -iu_y(x_0, y_0) + v_y(x_0, y_0) \tag{16}$$

According to the definition of derivative of complex function, the limit of horizontal direction equal to limit of vertical direction:

$$u_x(x_0, y_0) + iv_x(x_0, y_0) = -iu_y(x_0, y_0) + v_y(x_0, y_0) \tag{17}$$

Then we conclude that only when function satisfied $u_x = v_y, v_x = -u_y$, it has derivative.

3.3.3. Holomorphic Function. Let f has domain in S, where $S \to C$. The function is holomorphic if f is differentiable at z_0 in S, that is when

$$\lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} - w_0 \right| = 0,\ w_0 \in \mathrm{C} \tag{18}$$

3.4. Integral of Complex Function and Goursat's Theorem [10]

3.4.1. Integral. integral of complex function is distinct from real function. In real function, the function is integrated in the range $[a, b]$. However, for complex function, the integral is embodying on a closed curve γ.

Closed curve γ is parametrized curve, composed of $\gamma = x(t) + iy(t)$. When a complex function is integral along the curve γ, $\int_\gamma f(z)dz$, it should denote as:

$$\int_\gamma f(z)dz = \int_\gamma udx - vdy + i\int_\gamma vdx + udy = \int_a^b f(\gamma(t))\gamma^{'}(t)dt \tag{19}$$

For example: calculate the $\int_\gamma f(z) = z + \frac{1}{z}dz$, where $\gamma(t) = e^{it}(t\ from\ 0\ to\ 2\pi)$

Answer: Because $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$

We get

$$\int_\gamma f(z)dz = i\int_0^{2\pi} (e^{it} + e^{-it})e^{it}dt = i\int_0^{2\pi} e^{2it}dt = 0 \tag{20}$$

3.4.2. Cauchy-Goursat Theorem [4]. If integral of f which is holomorphic in a simply connected region D, then $f(z)$ in D has integral along arbitrary curve C:

$$\int_c f(z)dz = 0 \tag{21}$$

Proof: $f(z) = u(x, y) + iv(x, y)$

$$\int_c f(z)dz = \int_C u(x, y) + iv(x, y)dxdy = \int_C udx - vdy + i\int_C vdx + udy \tag{22}$$

$$= \iint_D u(x, y) + iv(x, y)dxdy$$

According to the Green's formula, we get:

$$\iint_D u(x, y) + iv(x, y)dxdy = \iint_D (\frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \frac{\partial \mathrm{u}}{\partial \mathrm{y}})dxdy + i\iint_D (\frac{\partial \mathrm{u}}{\partial \mathrm{x}} - \frac{\partial \mathrm{v}}{\partial \mathrm{y}})dxdy \tag{23}$$

Then according to the Cauchy-Riemann Equation:

DOI: 10.54254/2753-8818/2/20220084

$$u_x(x_0, y_0) + iv_x(x_0, y_0) = -iu_y(x_0, y_0) + v_y(x_0, y_0) \tag{24}$$

Or in polar form:

$$\frac{\partial \mathrm{u}}{\partial \mathrm{x}} = \frac{\partial \mathrm{v}}{\partial \mathrm{y}}, \ \frac{\partial \mathrm{u}}{\partial \mathrm{y}} = -\frac{\partial \mathrm{v}}{\partial \mathrm{x}} \tag{25}$$

We conclude that the integral

$$\iint_D (\frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \frac{\partial \mathrm{u}}{\partial \mathrm{y}}) dxdy + i \iint_D (\frac{\partial \mathrm{u}}{\partial \mathrm{x}} - \frac{\partial \mathrm{v}}{\partial \mathrm{y}}) dxdy = 0 \tag{26}$$

Thus,

$$\int_c f(z)dz = 0 \tag{27}$$

3.4.3. The Closed-circuit Deformation Theorem and Compound Closed-circuit Theorem. First is the Closed-circuit Deformation Theorem. Let a domain D has boundary C_0 which is a simple closed curve. The function $f(z)$ is analytic along the closed curve in the region D, getting C-G Equation:

$$\int_c f(z)dz = 0 \tag{28}$$

Now, another simple closed curve C_1 is in the D. The value of $\int_c f(z)dz = 0$ does not change when the closed curve is added in the region if the curve does not pass through the points that are not analytic.

Proof: making a guide AB to connect the C_0 and C_1(a closed curve C_0 includes another curve C_1, with a guide line connects two curves at point A on C_0 and B on C_1, and C_1 should be calculated in negative direction). According to the C-G Equation, we get that:

$$\int_{AB} f(z)dz + \int_{C_0} f(z)dz + \int_{BA} f(z)dz + \int_{C_1^-} f(z)dz = 0 \tag{29}$$

Because

$$\int_{AB} f(z)dz + \int_{BA} f(z)dz = 0 \tag{30}$$

Get

$$\int_{C_0} f(z)dz + \int_{C_1^-} f(z)dz = 0 \tag{31}$$

Thus,

$$\int_{C_0} f(z)dz = \int_{C_1} f(z)dz \tag{32}$$

Compound Closed-circuit Theorem states that a simple closed curve C_0 is in the domain D, $C_1, C_2, \ldots, C_n$ are simple closed curves in the C_0, those curves do not include or intersect with each other, and the regions bounded by curves are all included in D, if $f(z)$ is analytic in D
, that

$$\int_{C_0} f(z)dz = \sum_{i=1}^{n} \int_{C_i} f(z)dz \tag{33}$$

Proof: Assuming that $C_0, C_1, \ldots, C_n$ are $n+1$ simple closed curves, and $C_i (i = 1,2, \ldots, n)$ are all concluded in C_0, domain D comprises these $n+1$ curves, let C becoming the boundary of D.

Known that $f(z)$ is analytic in domain D, and is continuous on C, some guides are made to connect $C_i(i = 1,2, \dots, n)$ and C_0, the D is divided into several small simple connected regions, $f(z)$ is analytic in these simply connected regions, and is continuous along the boundary. Using the Cauchy-Goursat theorem, the

$$\int_c f(z)dz = 0 \tag{34}$$

Which is that the integral on the boundary of each simply connected region is equal to 0, and the sum of the integrals on the boundaries of all the simply connected regions are also 0.

We should notice that the integrals on the guides are been calculated by both sequentially and reversely two times, that is their upper bound and lower bound have been changed during two calculations of integral, offsetting each other to let the integral on the guides are 0. The integral of boundaries of all the above single connected regions minus the integral on the guides, which is equal to the integral on the boundary C of the original domain D, that is $\int_C f(z)dz = 0$.

And for C_0, the integral calculated in the positive direction, and for the $C_i(i = 1,2, \dots, n)$, the integral calculated in the negative direction (if all of simple closed curves have been connected, the inner curves $C_i(i = 1,2, \dots, n)$ are always been integral in reverse direction with outer curve), so we get

$$\int_C f(z)dz = \int_{C_0} f(z)dz + \int_{C_1^-} f(z)dz + \int_{C_2^-} f(z)dz + \cdots + \int_{C_n^-} f(z)dz$$
$$= \int_{C_0} f(z)dz + \sum_{i=1}^{n} \int_{C_i^-} f(z)dz = 0 \tag{35}$$

By moving the term, and according to

$$\int_c f(z)dz = 0 \tag{36}$$

We get

$$\int_{C_0} f(z)dz = \sum_{i=1}^{n} \int_{C_i} f(z)dz \tag{37}$$

4. An Identity Involving Complex Number

*4.1. The Theorem of **nth** Roots of Unity of Complex Number*

The equation $z^n = 1$, where $n \in N^*$, must have n different roots

$$\hat{z}_k = \sqrt[n]{1} = cos\frac{2\pi k}{n} + i\,sin\frac{2\pi k}{n}, k = 0,1, \dots, n-1 \tag{38}$$

Which are called the $n - th$ roots of unity.

Generally, for the equation

$$z^n = w \tag{39}$$

Where $w = r(cos\theta + i\,sin\theta)$, the roots are

$$\hat{z}_k = \sqrt[n]{w} = \sqrt[n]{r}\left(cos\frac{\theta + 2\pi k}{n} + i\,sin\frac{\theta + 2\pi k}{n}\right), k = 0,1, \dots, n-1 \tag{40}$$

Where θ is any argument of w.

It follows that

$$z_k = \sqrt[n]{w} = \sqrt[n]{r}\left(cos\frac{\arg w}{n} + i\,sin\frac{\arg w}{n}\right)\hat{z}_k, k = 0,1, \dots, n-1 \tag{41}$$

And

DOI: 10.54254/2753-8818/2/20220084

$$z_k = \sqrt[n]{r} e^{i\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right)}, k = 0,1, \ldots, n-1 \tag{42}$$

For example, calculating the $z^3 = i$

Answer: we know that:

$$z^3 = e^{i\frac{\pi}{2}} \tag{43}$$

So, we get

$$z = e^{i\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right)}, k = 0,1, \ldots, n-1 \quad z = e^{i\frac{\pi}{6}}, e^{i\frac{5\pi}{6}}, e^{i\frac{3\pi}{2}} \tag{44}$$

People put this theorem in geometry. When a complex number $z^n = x$, where $n \in N^*$. The all of roots divided complex plane into n parts. z is corresponding to the angle θ, and other roots correspond to $\theta + \frac{2\pi k}{n}$. The angle between two roots are equals to the $\frac{2\pi}{n}$.

If the coefficient of equation are real numbers, we can find that the distribution of roots is symmetrical about $x - axis$. This indicate that a complex root is always appears with its conjugate pair. And this point far away $\sqrt[n]{r}$ from the original point.

The Vieta's Theorem is also used in the complex polynomials. A polynomial

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0 \tag{45}$$

The polynomial has roots $z_i (i = 1,2, \ldots, n)$

We can get

$$\frac{a_0}{a_n} = (-1)^n \prod_{k=1} z_k \tag{46}$$

$$\frac{a_1}{a_n} = (-1)^{n-1} \sum_{j=1}^{n} \prod_{k \neq j} z_k \tag{47}$$

$$\cdots$$

$$\frac{a_{n-1}}{a_n} = -\sum_{k=1}^{n} z_k \tag{48}$$

For example: Let's calculating the roots of $\sqrt[n]{1+i}$.

Answer: Let $z = 1 + i$. Then, $z = 1 + i = \sqrt{2}\left(cos\frac{\pi}{4} + i\,sin\frac{\pi}{4}\right)$.

Thus $\sqrt[n]{1+i} = \sqrt[n]{z} = \sqrt[2n]{2}(cos\frac{2k\pi+\frac{\pi}{4}}{n} + i\,sin\frac{2k\pi+\frac{\pi}{4}}{n})(k = 0,1,2, \ldots, n-1)$

4.2. Primitive Unit Root and Cyclotomic Polynomial

From the former statement, we know that $cos\frac{2\pi}{n} + i\,sin\frac{2\pi}{n}$ is a unit root of $z^n - 1 = 0$, and $cos\frac{2\pi k}{n} + i\,sin\frac{2\pi k}{n}, k = 0,1, \ldots, n-1$ are all complex unit roots of $z^n - 1 = 0$, the $n - th$ roots of unit. The roots $z_k (k = 0,1,2, \ldots, n-1)$, when k and n are coprime integers, $k \perp$ n, we call that z_k is the primitive unit root. We use $\varphi(n)$ to denote the number of primitive unit root.

A property of primitive unit root: $(z_k)^n = 1$.

Cyclotomic Polynomial is a polynomial that composed of primitive unit roots. The definition is that

$$\Phi_n(x) = \prod_{k \perp n \cup 0 \leq k \leq n-1} (x - z_k) \tag{49}$$

Among it, $\Phi_n(x)$ is the signal of cyclotomic polynomial.

The Cyclotomic Polynomial is the factor of $z^n - 1 = 0$. To be specific, we let $F(z) = z^n - 1$, and $\Phi_n(x) | F(z)$.

DOI: 10.54254/2753-8818/2/20220084

For example: stating the primitive unit roots of $F(z) = z^6 - 1$, and the cyclotomic polynomial of $F(z)$.

Answer: there are five roots of the polynomial, which are $z^0, z^1, z^2, z^3, z^4, z^5$. From the definition of primitive unit root, we know that it has two primitive unit root that are z^1, z^5. And the cyclotomic polynomial of it is $\Phi_6(x) = \prod(x - z^k)(k = 1,5)$. The $\Phi_6(x) = (x - z^1)(x - z^5)$. We should calculate z^1 and z^5, the answer equal to $e^{i\frac{\pi}{3}}$ and $e^{i\frac{5\pi}{3}}$. Put these two values into the polynomial, we get $\Phi_6(x) = x^2 - x(e^{i\frac{\pi}{3}} + e^{i\frac{5\pi}{3}}) + e^{i\frac{\pi}{3}}e^{i\frac{5\pi}{3}} = x^2 - x + 1$.

Finally, we get the expression of cyclotomic polynomial that is

$$\Phi_6(x) = x^2 - x + 1 \tag{50}$$

4.3. An Identity Involving Complex Number

Assume that the roots of the equation $x^n = 1, n \geq 2, n \in N$ are 1, z_1, $z_2, \dots, z_{n-1}$. Then, we have $\frac{1}{1-z_1} + \frac{1}{1-z_2} + \cdots + \frac{1}{1-z_{n-1}} = \frac{n-1}{2}$.

Proof: $\because$ $z_1, z_2, \dots, z_{n-1}$ are the roots of $z^n = 1 (n \geq 2, n \in N)$,

$$\therefore\ z_k = cos\frac{2\pi k}{n} + i\ sin\frac{2\pi k}{n} (k = 0,1,2, \dots, n-1). \tag{51}$$

(According to the statement mentioned in the former place, that each root is corresponding to a particular angle)

And

$$\frac{1}{1 - z_k} = \frac{1}{\left(1 - cos\frac{2\pi k}{n}\right) - i\ sin\frac{2\pi k}{n}} = \frac{1}{2sin\frac{\pi k}{n}(sin\frac{\pi k}{n} - i\ cos\frac{\pi k}{n})}$$

$$= \frac{sin\frac{\pi k}{n} + i\ cos\frac{\pi k}{n}}{2sin\frac{\pi k}{n}} (k = 0,1,2, \dots, n-1)$$

$$= \frac{1}{1 - z_1} + \frac{1}{1 - z_2} + \cdots + \frac{1}{1 - z_{n-1}} = \sum_{k=1}^{n-1}(\frac{1}{2} + \frac{1}{2}icot\frac{\pi k}{n}). \tag{52}$$

(Notice that for all integer n, $\sum_{k=1}^{n-1} cot\frac{\pi k}{n} = 0$, this is because $cot\frac{\pi k}{n}$ can be denoted as $\frac{sin\frac{\pi k}{n}}{cos\frac{\pi k}{n}}$. We transform the formula to exponential form: $\frac{e^{ki\frac{\pi}{n}} - e^{ki\frac{\pi}{n}}}{e^{ki\frac{\pi}{n}} + e^{ki\frac{\pi}{n}}}$. In it, k is the only changing number that should be added together, and we can calculate it as a geometric sequence.)

We can only calculate $\sum_{k=1}^{n-1}\frac{1}{2} = \frac{n-1}{2}$,

$$\therefore \frac{1}{1-z_1} + \frac{1}{1-z_2} + \cdots + \frac{1}{1-z_{n-1}} = \frac{n-1}{2}. \tag{53}$$

5. Conclusion

From long time, people had realized that $\sqrt{-1}$ does not meaningless. From the formula to the square root of negative number, people found proof and numeric meaning of complex number. They know that, number does not merely exist in apparent part. Mathematicians who base on the understanding of complex number have developed intricate calculation of complex number. When this field that expands the number system from real number to imaginary number has been perfected gradually, when more and more scholars want to step into research of complex number, the investment of complex number is going to mature. An advanced subject can be used in multifaced application. In modern days, we can say that complex number is not only used in pure math field, but also be helpful in other subjects including

DOI: 10.54254/2753-8818/2/20220084

electronics and quantum mechanics. One day in the future, complex number will be used in further region and more other subjects.

References

[1] Ahlfors, L. (1966) Complex analysis. MGH

[2] Berlinghoff, W. P. and Gouvea, F. Q. (2015) Math through the ages Vol32. Farmington, Oxton House Publish.

[3] Brown, J. W. and Churchill, R. V. (2009) Complex Variables and Applications. McGraw-Hill.

[4] Calderon, A. P. (1958) Uniqueness in the Cauchy problem for partial differential equations. The Johns Hopkins University Press.

[5] Dyn'kin, E.M. (1972) Investigations in Linear Operators and Function Theory, 128-131.

[6] Elias M. Stein. (2003) Complex analysis Vol.2. Princeton University Press, 1-37.

[7] Fisher, S. D. (1999) Complex Variables. Dover Publication, 171-183.

[8] Gelbaum, R.B. (1992) Problems in real and complex analysis. Springer-Verlag.

[9] Hahn, L.S. and Epstein, B. (1996) Classical complex analysis. Jones and Bartlett Publishers

[10] Moore, E. H. (1900) Transactions of the American Mathematical Society. American Mathematics Society, 499-506