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# Majorana fermions and its application on topological quantum computer

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**Abstract.** The majorana fermion represents a kind of particle which is its own antiparticle. This paper aims to analyze the majorana fermions from theoretical aspect and application aspect, including the derivation of Dirac equation and the practicality of topological quantum computer. This paper explores the idea of Majorana fermions from Dirac equation and the anyons which is an important quasi-particle to build a topological quantum computer. After that, the potentials of quantum computer are emphasized as well. Several current major difficulties faced by building a quantum computer have also been discussed, including the decoherence of qubit and the errors during the operation of qubit. The advantages of topological computer are mentioned as well, especially the high resistance to local perturbation. The important property of two-dimensional non-abelian anyons has been discussed as well. Finally, the important relation of majorana fermions with anyons are introduced and the reason why majorana fermions is important has also been revealed.

**Keywords:** Dirac equation, majorana fermions, topological quantum computer, non-abelian anyons.

## 1. Introduction

In the vast landscape of quantum mechanics, one of the most intriguing discoveries is the Majorana fermion. Proposed by Majorana in 1937, this particle possesses a unique quality: it is its own antiparticle [1]. This self-conjugate nature enables them to maintain coherence against local perturbations and decoherence [2]. These properties position Majorana fermions as a cornerstone for developing topological quantum computers, offering a pathway to fault-tolerant quantum computation [3].

Quantum computing stands at the technological frontier, which can revolutionize myriad fields by solving problems deemed insurmountable for classical computers. The potential of quantum computers extends across cryptography, drug discovery, optimization problems [4], artificial intelligence [5], and beyond. However, Quantum computers rely on maintaining the quantum states of qubits, which are inherently fragile and sensitive to their surroundings. Interaction with the environment can cause quantum decoherence, leading to the loss of quantum information. Quantum noise from the environment can also introduce errors in calculations [6]. Nevertheless, the integration of Majorana fermions can perfectly solve this problem by mitigating decoherence and enhancing fault tolerance, thereby contributing to the realization of scalable and practical quantum computing systems.

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This paper explores the theoretical foundations and potential applications of Majorana fermions in advancing the field of quantum information science, from the derivation of Dirac equation to the majorana fermions in condensed matter physics.

## 2. Dirac equation and Majorana fermion

### *2.1. From Schrödinger equation to Dirac equation*

The Schrödinger equation was proposed by Erwin Schrödinger in 1926. It is fundamental to quantum mechanics and must be understood to further understand the derivation of the Majorana fermions. However, when people applying the particle with relativistic frame work, there are several problems emerging within the Schrödinger equation. Since the Schrödinger equation was not designed to fit with relativistic framework at the beginning, it is reasonable that there are some problems using the equation on relativistic particles. To solve these problems, there are several 20th century physicists to make the Schrödinger equation compatible with the special relativity. Dirac equation formulated by Paul Dirac in 1928 perfectly solves these problems.

From the special relativity proposed by the Albert Einstein in 1905, space and time are not independent but are intertwined and form a four-dimensional continuum known as spacetime. It is found that

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t}, \quad (1)$$

where $m$ is the mass, $t$ is time, $\hbar$ is Planck's constant over $2\pi$, and $\psi$ is the wavefunction of the system. However, in the Schrödinger equation, the space derivative is second order on the left, while the time derivative is first order on the right. That is inconsistent with the spacetime concept, which means the Schrödinger equation cannot be used for relativistic particles.

The Klein-Gordon equation was first introduced in the context of quantum mechanics by Oskar Klein and Walter Gordon in 1926-1927. It was one of the early attempts to unite quantum mechanics with special relativity and describes spin-0 particles, both free and interacting, in a relativistic framework. For a particle of mass m and energy E, the relativistic energy-momentum relation in special relativity is given by

$$E^2 = \left(mc^2\right)^2 + (pc)^2, \quad (2)$$

where $E$ is the energy, $m$ is the mass, $p$ is the momentum, and $c$ is the speed of light.

In quantum mechanics, energy and momentum are represented by operators. The energy operator is $\hat{E} = i\bar{h}\frac{\partial}{\partial t}$ and the momentum operator is $\hat{p} = -i\bar{h}\nabla$. By substituting the energy and momentum operator into the energy-momentum relation, the Klein-Gordon equation can be shown:

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial x^2} - \nabla^2\psi + \frac{m^2c^2}{\bar{h}^2} = 0, \quad (3)$$

where $m$ is the mass, x is the displacement, $c$ is the speed of light, $\hbar$ is Planck's constant over $2\pi$, and $\psi$ is the wavefunction of the system.

The Klein-Gordon equation satisfies the requirement of relativity since it has the double derivative with time and space. However, there is other problems with this equation. Since the Klein-Gordon equation treat both space and time with secondary derivative, that means single initial condition cannot define the unique wavefunction in order to find the solution. That will offer the probability of negative energy and thus gives negative probability density since the probability density is proportional to energy, which is unreasonable.

To solve the problem with Klein-Gordon equation, Dirac seek to make the differential equation into first order, which thus can give certain solution with one initial condition [7]. That thus can eliminate the solution of negative energy, which offers the negative probability density. Similar with the Klein-Gordon equation, the relativistic energy-momentum relation in special relativity is given by

$$E^2 = \left(mc^2\right)^2 + (pc)^2. \tag{4}$$

To make this equation into first order equation, the square root of the equation must be known. Thus,

$$p^2c^2 + m^2c^4 = \left(c\vec{\alpha}\vec{p} + \beta mc^2\right)^2 = \left(c\alpha_1 p_1 + c\alpha_2 p_2 + c\alpha_3 p_3 + \beta mc^2\right)^2 \tag{5}$$

To satisfy this equation, there are three conditions that $\vec{\alpha}$ and $\vec{p}$ must satisfy. The first condition is that when each term is squared, there should not be any $\alpha$ or $\beta$, which thus gives the first condition

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1. \tag{6}$$

The second and third conditions are that every cross term when expanding the square of the equation must also be cancelled, which gives the second and third conditions. That is, $\alpha_i\alpha_j + \alpha_j\alpha_i = 0$ for $i \neq j$, and $\alpha_i\beta + \beta\alpha_i = 0$. Since there is not any number that can satisfy the second and third condition, Dirac found that $\vec{\alpha}$ and $\beta$ are four by four matrices [7]

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{7}$$

By summarizing the equations, the square root of energy-momentum equation can be given by $E^2 = (c\vec{\alpha}\vec{p} + \beta mc^2)^2$ and $E = c\vec{\alpha}\vec{p} + \beta mc^2$. By substituting the energy operator and momentum operator into the equation, the Dirac equation can be given by $-i\bar{h}\frac{\partial}{\partial t}\psi = (c\vec{\alpha}\vec{p} + \beta mc^2)\psi$. By defining the $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha_i$, the Dirac equation can be simplified as following:

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0. \tag{8}$$

### 2.2. Majorna fermions

In particle physics, every elementary particle has a corresponding antiparticle, which is a particle that has the same mass but opposite charge and other quantum numbers. These include electron and positron, proton and antiproton. When a particle and its antiparticle meet, they can annihilate each other, resulting in the release of energy according to Einstein's mass-energy equivalence principle, $E = mc^2$ [2]. Before the idea of the majorana fermions, people mostly believed that the particle is completely different with antiparticles. However, Ettore Majorana in 1937 suggested that there could be a real wavefunction to the solution of Dirac equation, which describes a fermion with only two degrees of freedom (as opposed to four for a Dirac fermion), which is neutral (no electric charge) and is its own antiparticle. That means there is a type of fermion that is its own antiparticle.

## 3. Topological quantum computer

### 3.1. Quantum computer

Before the discussion of quantum computer, the principal difference between the conventional physics and quantum physics must be understand thoroughly. Consider the two-slit experiment, when electrons are fired one by one toward a barrier with two slits, the electron should one slit or another according to conventional physics. However, in quantum physics, each electron goes through both slits in a supperposition and interfere itself, producing the interference pattern in the detector screen. This interference pattern is a direct manifestation of the electron taking multiple trajectories and their coherent sum (interference) determining the final state. So instead of calculating one path a time in conventional computer, the quantum computer can take multiple path parallelly and determine its state by their coherent sum. That property give enormous advantages toward quantum computer comparing with conventional computer, including decryption and understanding of complex materials.

The basic computation model for quantum computer can be simplified as three step: initialization, unitary evolution, and measurement [6]. Suppose that there is a system in Hilbert space $H$, the initial state of the system can be demonstrated by $|\psi_0>$. The system then will evolve to some final state

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$U(t)\ |\psi_0>$, according to the the Schrödinger equation. The Hamiltonian of this evolution must be controlled so that the final state $U(t)$ can be obtained by any unitary transformation. The last step is to make measurement [6]. The initialization correspond to the input, while the measurement is the output. The unitary evolution is the software program to run.

Corresponding to the bit in conventional computer, which represent 0 or 1, qubit is the fundamental part of quantum computer, which is a quantum two-state system. According to the superposition principle, the qubit ( two-state quantum system) can be a infinitely many superposition of 0 and 1 ( $a|0>+b\,|1>$ ) [6].

The errors is the major problem to build a quantum computer. In conventional computer, the errors can always be corrected through keeping the copies of data and checking it with the copies in the intermediate stage. However, checking the information in the intermediate stage of calculation in quantum computer cannot work. The measurement in the quantum computer in the intermediate stage will collapse the quantum system, which will cause the loss of information [6]. To solve that, the error correction is proposed in 1995. The basic idea is that instead of direct observation toward the quantum state, the error can be detected by representing information redundantly. However, during the error corrections protocols, there will be more errors occurring, which make the error correction protocol unclear to be the solution toward the problem of quantum computer [6].

### *3.2. Non-Abelian anyons*

In condensed matter physics, quasi-particles are emergent phenomena that arise from the collective behavior of interacting particles in a material.[8] Anyons are theoretical quasiparticles that exist in two-dimensional systems and have properties distinct from the familiar fermions and bosons [9]. To understand the special property of anyons, the property of fermions and boson must be understand first. The wavefunction for a system of identical fermions is anti-symmetric, which means that exchanging the quantum states of two fermions, the wavefunction changes sign, $|\psi>\to e^{i\pi}|\psi>$. The wavefunction for a system of identical bosons is symmetric, which means that exchanging the quantum states of two bosons does not change the wavefunction, $|\psi>\to|\psi>$.

However, the swapping of identical anyons can lead to a phase factor that is neither 0 (as for bosons) nor $\pi$ (as for fermions), but can take any value in between, $|\psi>\to e^{i\theta}|\psi>\ \ (0\le\theta\le\pi)$. Abelian means the order of swapping operation doesn't matter. For the non-Abelian anyons, they have a special property that when braiding the anyons (moving one anyon around another in two-dimensional space), the system's quantum state undergoes a transformation. That means the information is no longer stored in the local properties, such as spin orientation of electron, but stored in the system [6].

### *3.3. Anyons in topological quantum computer*

Process of braiding non-Abelian anyons can be thought of as enacting quantum gates on the system [10]. Each distinct braiding sequence corresponds to a different quantum gate. Since the information is stored in the global properties of the system (its topology) and not in local properties, it is highly resistant to local errors. The main advantage of using non-Abelian anyons for quantum computation is the inherent error resistance provided by their topological nature. In a topological quantum computer, errors that typically afflict quantum systems (like phase errors or bit flips) need to act in a coordinated, non-local manner to cause a logical error, which is highly improbable. After all, Majorana fermions are the simplest non-abelian anyons that exhibit the unique braiding statistics. To actually create practical quantum computer, the majorana fermions is the key.

## 4. Conclusion

In conclusion, quantum computer has been developed rapidly in recent decades. However, several engineering problems has also been found in this process. The practicality of quantum computer has been questioned during last decades. However, with the deeper understanding of quantum physics, topological quantum computer, which can resist the local perturbations, has been proposed. In conventional quantum computer, the errors caused by the local perturbation is extremely difficult to

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correct because the direction observation of qubit will make the wavefunction collapse, which will cause the loss of information. However, by braiding the non-abelian anyons, topological quantum computer can store the information in the global property of quantum system, instead of local property, which can give it high resistant to local error. Eventually, the problem of topological quantum computer becomes finding the non-abelian anyons. Majorana fermions, the particle that is the antiparticle of itself, is actually one of non-abelian anyons. That makes finding the majorana fermions extremely important to topological quantum computer.

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